

# On Mean Field Convergence and Stationary Regime

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## Abstract

Assume that a family of stochastic processes on some Polish space  $E$  converges to a deterministic process; the convergence is in distribution (hence in probability) at every fixed point in time. This assumption holds for a large family of processes, among which many mean field interaction models and is weaker than previously assumed. We show that any limit point of an invariant probability of the stochastic process is an invariant probability of the deterministic process. The results are valid in discrete and in continuous time.

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## 1. Introduction

This paper is motivated by results on mean field interaction models and stochastic approximation algorithms which obtain convergence of a family of stochastic processes to a deterministic limit (Kurtz, 1970; Sandholm, 2006; Bordenave et al., 2007; Graham and Méléard, 1994; Benaïm and Le Boudec, 2008). Often, convergence is over finite time horizons, which asks the question of whether the convergence extends to the stationary regime. In this paper, we show that some form of convergence of the stationary regimes follows systematically from convergence at any fixed and finite time horizon, under very weak assumptions. Previous answers to this question exist with stronger assumptions than here; for example, the space is finite dimensional and the deterministic limit is a dissipative ODE (Benaïm, 1998), or the set of invariant distributions is tight (Fort and Pages, 1999). Such assumptions cannot always be made, consider for example the cases in (Chaintreau et al., 2009; Bordenave et al., 2007); our result requires much weaker assumptions and appears to be more general.

More precisely, we consider a family of stochastic processes  $Y^N$ , indexed by  $N = 1, 2, \dots$  over some Polish space  $E$ ; we assume that the processes have the property that, as  $N \rightarrow \infty$  and the initial conditions  $y^N(0) \rightarrow y(0)$ , the marginals of the process  $Y^N(t)$  converge in distribution to some deterministic  $y(t)$ , where convergence is at every fixed time  $t$  (see Hypothesis 1 below). We show that, under the (mild) assumption that the processes are Feller, this is sufficient to obtain that any limit point of an invariant probability of  $Y^N$  is an invariant probability of the deterministic process (and thus its support is included in its Birkhoff center). Note that we do not need to assume any semi-flow nor continuity property for the limiting deterministic process.

In the special case where the deterministic process has a unique limit point  $y^* = \lim_{t \rightarrow \infty} y(t)$  and where the sequence of invariant probabilities  $\Pi^N$  is tight, it follows imme-

diately that  $\Pi^N$  converges to the Dirac mass at  $y^*$ . This result is known in the context of stochastic approximation algorithms; our results here extend it to a more general setting.

Our result is also more general as it applies to other cases. Mean field interaction models were often used as practical approximations of complex interacting object systems, where the stationary distribution of the system  $Y^N$  is approximated by the stationary regime of an ordinary differential equation (ODE); this was applied for example to TCP connections (Tinnakornsrisuphap and Makowski, 2003; Baccelli et al., 2006; Graham and Robert, 2009), HTTP flows (Baccelli et al., 2004), bandwidth sharing between streaming and file transfers (Kumar and Massoulié, 2007), mobile networks (Chaintreau et al., 2009), robot swarms (Martinoli and Easton, 2002), transportation systems (Afanassieva et al., 1997), reputation systems (Le Boudec et al., 2007), just to name a few. Previous results are obtained when the ODE has a unique limit point to which all trajectories converge. Not only does our result extends this finding to more general spaces, it also extends it to the cases where there is not a unique limit point. For example, in (Bordenave et al., 2007), the ODE has a unique limit point under some restrictive assumptions on the model parameters; if these assumptions do not hold, the ODE may have limit cycles, as shown in (Cho et al., 2010), and nothing can be concluded from (Bordenave et al., 2007). Using our results, it follows that the limit points of any invariant probability has a support included in the limit cycles.

## 2. Assumptions and Notation

### 2.1. A Collection of Random Processes

Let  $(E, d)$  be a Polish space and  $\mathcal{P}(E)$  the set of probability measures on  $E$ , endowed with the topology of weak convergence. Let  $\mathcal{C}_b(E)$  be the set of bounded continuous functions from  $E$  to  $\mathbb{R}$ .

We are given a collection of probability spaces  $(\Omega^N, \mathcal{F}^N, \mathbb{P}^N)$  indexed by  $N = 1, 2, 3, \dots$  and for every  $N$  we have a process  $Y^N$  defined on  $(\Omega^N, \mathcal{F}^N, \mathbb{P}^N)$ . Time is either discrete or continuous. In the discrete time case,  $Y^N(t)$  is a collection of random variables indexed by  $N = 1, 2, 3, \dots$  and  $t \in \mathbb{N}$ . In the continuous time case, let  $D_E[0, \infty)$  be the set of càdlàg functions  $[0, \infty) \rightarrow E$ ;  $Y^N$  is then a stochastic process with sample paths in  $D_E[0, \infty)$ .

We denote by  $Y^N(t)$  the random value of  $Y^N$  at time  $t \geq 0$ . Let  $E^N \subset E$  be the support of  $Y^N(0)$ , so that  $\mathbb{P}^N(Y^N(0) \in E^N) = 1$ .

We assume that, for every  $N$ , the process  $Y^N$  is Feller, in the sense that for every  $t \geq 0$  and  $h \in \mathcal{C}_b(E)$ ,  $\mathbb{E}^N [h(Y^N(t)) | Y^N(0) = y_0]$  is a continuous function of  $y_0 \in E$ .

**Definition 1.** A probability  $\Pi^N \in \mathcal{P}(E)$  is invariant for  $Y^N$  if  $\Pi^N(E^N) = 1$  and for every  $h \in \mathcal{C}_b(E)$  and every  $t \geq 0$ :

$$\int_E \mathbb{E}^N [h(Y^N(t)) | Y^N(0) = y] \Pi^N(dy) = \int_E h(y) \Pi^N(dy)$$

## 2.2. A Deterministic Measurable Process

Further, let  $\varphi$  be a deterministic process, i.e. a mapping

$$\begin{aligned}\varphi : \mathbb{T} \times E &\rightarrow E \\ t, y_0 &\mapsto \varphi_t(y_0)\end{aligned}$$

where  $\mathbb{T} = \mathbb{N}$  or  $\mathbb{T} = [0, \infty)$ .

We assume that  $\varphi_t$  is measurable for every fixed time  $t \geq 0$ . Note that there is no assumption here that  $\varphi$  is continuous nor that it is a flow.

**Definition 2.** A probability  $\Pi \in \mathcal{P}(E)$  is invariant for  $\varphi$  if for every  $h \in \mathcal{C}_b(E)$  and every  $t \geq 0$ :

$$\int_E h(\varphi_t(y)) \Pi(dy) = \int_E h(y) \Pi(dy)$$

## 2.3. Convergence Hypothesis

We assume that, for every fixed  $t$  the processes  $Y^N$  converge in distribution to the deterministic process  $\varphi$  as  $N \rightarrow \infty$  for every collection of converging initial conditions. More precisely:

**Hypothesis 1.** For every  $y_0$  in  $E$ , every sequence  $(y_0^N)_{N=1,2,\dots}$  such that  $y_0^N \in E^N$  and  $\lim_{N \rightarrow \infty} y_0^N = y_0$ , and every  $t \geq 0$ , the conditional law of  $Y^N(t)$  given  $Y^N(0) = y_0^N$  converges weakly to the Dirac mass at  $\varphi_t(y_0)$ . That is

$$\lim_{N \rightarrow \infty} \mathbb{E}^N [h(Y^N(t)) | Y^N(0) = y_0^N] = h \circ \varphi_t(y_0)$$

for all  $h \in \mathcal{C}_b(E)$  and any fixed  $t \geq 0$ .

## 2.4. Examples.

In discrete time,  $Y^N$  is a Markov chain on  $E^N$ , as in (Le Boudec et al., 2007), where  $E = \mathcal{P}(S)$  for some compact set  $S$ ,  $Y^N$  is the occupancy measure of a process on  $S$ , and  $E^N$  is the set of probabilities that are the sum of  $N$  Dirac masses. Here Definition 1 coincides with invariant probability for a Markov chain. The deterministic process is an iterated map, and Definition 2 coincides with invariant probability of an iterated map.

In continuous time,  $Y^N$  may be a Markov process on  $E^N$ , as in (Kurtz, 1970; Sandholm, 2006; Bordenave et al., 2007; Graham and Méléard, 1994; Benaïm and Le Boudec, 2008). Definition 1 coincides here with invariant probability for a Markov process. The deterministic process is a semi-flow, and Definition 2 coincides with invariant probability for semi-flows. If  $E$  is finite dimensional, the deterministic process is an ODE or a differential inclusion.

Still in continuous time,  $Y^N$  may also be the continuous linear interpolation of a discrete time process, as in (Bordenave et al., 2007; Benaïm and Le Boudec, 2008) (in this case it is not a Markov process). An invariant probability for  $Y^N$  is here an invariant probability of the interpolated Markov chain.

Hypothesis 1 holds in (Le Boudec et al., 2007; Kurtz, 1970; Sandholm, 2006; Bordenave et al., 2007; Benaïm and Le Boudec, 2008) as a consequence of stronger convergence results; for example in (Kurtz, 1970) there is almost sure, uniform convergence for all  $t \in [0, T]$ , for any  $T \geq 0$ .

### 3. Convergence of Invariant Probabilities

#### 3.1. Main Theorem

**Theorem 1.** *Assume Hypothesis 1 holds and let  $\Pi \in \mathcal{P}(E)$  be a limit point of the sequence  $\Pi^N$ , where  $\Pi^N$  is an invariant probability for  $Y^N$ . Then  $\Pi$  is an invariant probability for  $\varphi$ .*

**Proof.** Let  $N_k$  be a subsequence such that  $\lim_{k \rightarrow \infty} \Pi^{N_k} = \Pi$  in the weak topology on  $\mathcal{P}(E)$ . By Skorohod's representation theorem for Polish spaces (Ethier and Kurtz, 2005, Thm 1.8), there exists a common probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  on which some random variables  $X^k$  for  $k \in \mathbb{N}$  and  $X$  are defined such that

$$\begin{cases} \text{law of } X^k = \Pi^{N_k} \\ \text{law of } X = \Pi \\ X^k \rightarrow X \text{ } \mathbb{P} - \text{a.s.} \end{cases}$$

Fix some  $t \geq 0$  and  $h \in \mathcal{C}_b(E)$ , and define, for  $k \in \mathbb{N}$  and  $y \in E$

$$a^k(y) \stackrel{\text{def}}{=} \mathbb{E} (h(Y^{N_k}(t)) | Y^{N_k}(0) = y)$$

Since  $\Pi^{N_k}$  is invariant for  $Y^{N_k}$ :

$$\int_E a^k(y) \Pi^{N_k}(dy) = \int_E h(y) \Pi^{N_k}(dy) \quad (1)$$

Hypothesis 1 implies that  $\lim_{k \rightarrow \infty} a^k(x^k) = h(\varphi_t(x))$  for every sequence  $x^k$  such that  $x^k \in E^{N_k}$  and  $\lim_{k \rightarrow \infty} x^k = x \in E$ . Now  $X^k \in E^{N_k}$   $\mathbb{P}$ -almost surely, since the law of  $X^k$  is  $\Pi^{N_k}$  and  $\Pi^{N_k}$  is invariant for  $Y^{N_k}$ . Further,  $X^k \rightarrow X$   $\mathbb{P}$ -almost surely; thus

$$\lim_{k \rightarrow \infty} a^k(X^k) = h(\varphi_t(X)) \text{ } \mathbb{P} - \text{almost surely} \quad (2)$$

Now  $a^k(X^k) \leq \|h\|_\infty$  and, thus, by dominated convergence:

$$\lim_{k \rightarrow \infty} \mathbb{E} (a^k(X^k)) = \mathbb{E} (h(\varphi_t(X))) \quad (3)$$

Using Eq.(1):

$$\lim_{k \rightarrow \infty} \int_E h(y) \Pi^{N_k}(dy) = \int_E h(\varphi_t(y)) \Pi(dy) \quad (4)$$

and thus

$$\int_E h(y) \Pi(dy) = \int_E h(\varphi_t(y)) \Pi(dy) \quad (5)$$

This holds for any  $h \in \mathcal{C}_b(E)$  and  $t \geq 0$ , which shows that  $\Pi$  is invariant for  $\varphi$ .  $\square$

### 3.2. The Continuous Semi-Flow Case

Note that our assumptions on  $\varphi$  are very weak. We now make an additional assumption:

**Definition 3.** The deterministic process  $\varphi$  is a continuous semi-flow if

1.  $\varphi_0(y) = y$
2.  $\varphi_{s+t} = \varphi_s \circ \varphi_t$  for all nonnegative  $s$  and  $t$
3.  $\varphi_t(y)$  is continuous in  $t$  and  $y$

If  $\varphi$  is a continuous semi-flow, it follows from Poincaré's recurrence theorem (Mané and Levy, 1987) that the support of any limit point of  $\Pi^N$  is included in the closure of the recurrent set :

$$R(\varphi) = \{x \in E : \liminf_{t \rightarrow \infty} d(x, \varphi_t(x)) = 0\}.$$

In particular, if the semi-flow has a unique limit point, we have:

**Corollary 1.** Assume Hypothesis 1 holds and that  $\varphi$  is a continuous semi-flow. Let  $\Pi^N$  be a sequence of invariant probabilities for  $Y^N$ . Assume that

1. the sequence  $(\Pi^N)_{N=1,2,\dots}$  is tight;
2. there is some  $y^* \in E$  such that for all  $y \in E$ ,  $\lim_{t \rightarrow \infty} \varphi_t(y) = y^*$ .

It follows that the sequence  $\Pi^N$  converges weakly to the Dirac mass at  $y^*$ .

Recall that tightness means that for every  $\epsilon > 0$  there is some compact set  $K \subset E$  such that  $\Pi^N(K) \geq 1 - \epsilon$  for all  $N$ . If  $E$  is compact then  $(\Pi^N)_{N=1,2,\dots}$  is automatically tight.

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